# Bond Formulae Derivation

Chris Tham

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## 1 Notation

| *Symbol* | *Description* |
|---|---|
| $P$ | market value of the bond per \$100 face value |
| $\omega$ | number of coupon payments per year, normally 2 |
| $i$ | market yield divided by $100\omega$ (i.e. 12% becomes .06) |
| $v$ | $\frac{1}{1+i}$ |
| $f$ | number of days to next coupon payment |
| $d$ | number of days in between last and next coupon payments |
| $g$ | coupon payment per \$100 |
| $x$ | 1 if the bond is cum-interest, 0 otherwise |
| $n$ | number of coupon payment periods from next coupon payment till maturity |

## 2 Fair Price of a bond

This section will show a derivation of the RBA bond valuation formula for the pricing of domestic treasury bonds (and semi-government stock) with at least 2 coupon payments left. Bonds with maturity dates less than 1 full coupon payment period are discounted in the same manner as bills.

The cash flows arising from a bond comprise two components:

- The right to receive interest (coupon) payments - usually at six monthly intervals.
- The right to receive the face value of the bond at the expiration of its term (*redemption*).

Hence, to determine the present value of a bond, there are three steps:

1. **Calculate the present value of interest payments.**
   On each coupon payment date, $g$ is paid to the bond holder. The discount factor per coupon period is $v$, since $i$ is the yield that we expect from holding the bond for exactly one coupon period. If we let $j = 0, 1, ..., n$ to represent the $n + 1$ coupons due to the bond holder in the future, then the value of the $j$-th coupon payment discounted back to the next coupon payment date is $gv^j$. However, there are $f$ days to the next coupon payment date, hence, the present value of the $j$-th coupon payment is $gv^j$ multiplied

by the discount factor $v^{f/d}$ or $gv^{j+f/d}$. Hence, the total of the present value of coupon payments is

$$\sum_{j=0}^{n} gv^{j+f/d}$$

There is an added complication in that the next coupon payment is not owed to the bond holder if the bond is purchased within two weeks of the next coupon payment date (ex-interest bond). More properly we should sum from $1-x$ to $n$ instead of 0 to $n$ so that $j = 0, 1, ..., n$ for cum-interest bonds and $j = 1, 2, ..., n$ for ex-interest bonds.

2. **Calculate the present value of the face value.**
The present value of the redemption value of the bond ($100) is simply the redemption value discounted back to the settlement date, or,

$$100v^{n+f/d}$$

3. **Add the two present values.**
Hence, a fair value for the price of bond is

$$\begin{aligned} P &= \left( \sum_{j=1-x}^{n} gv^{j+f/d} \right) + 100v^{n+f/d} && (1) \\ &= v^{f/d} \left[ g \left( \sum_{j=1-x}^{n} v^{j} \right) + 100v^{n} \right] && (2) \end{aligned}$$

To derive the RBA bond valuation formula from our formula, we define

$$S \equiv \sum_{j=1}^{n} v^{j} \tag{3}$$

Then,

$$\begin{aligned} S &= v + v^2 + v^3 + ... + v^n \\ vS &= v^2 + v^3 + v^4 + ... + v^{n+1} \\ (1-v)S &= v - v^{n+1} \\ S &= \frac{v - v^{n+1}}{1-v} \end{aligned}$$

But

$$\frac{1}{1-v} = \frac{1}{1-\frac{1}{1+i}} = \frac{1+i}{1+i-1} = \frac{1+i}{i} = \frac{1}{iv}$$

So

$$S = \frac{v - v^{n+1}}{iv} = \frac{1-v^n}{i} \tag{4}$$

and we observe [1] that

$$\sum_{j=1-x}^{n} gv^j = x + \frac{1-v^n}{i} \tag{5}$$

By substituting (5) into (2), we obtain the standard RBA bond valuation formula

$$P = v^{f/d}\left[g\left(x + \frac{1-v^n}{i}\right) + 100v^n\right] \tag{6}$$

## 3 Formula for Bond Duration

The duration of an asset is the mean term of all future cash flows, weighted by present value. In the case of a bond, it is equal to the sum of present values of the interest payments and redemption value multiplied by the years to maturity of each cash flow.

The $j$-th coupon payment has a term of $\frac{j+f/d}{\omega}$ years, with present value $gv^{j+f/d}$. The redemption value has a term of $\frac{n+f/d}{\omega}$ years, with present value $100v^{n+f/d}$.

Hence, the duration of a bond can be expressed as

$$\begin{aligned} D &\equiv \frac{1}{P}\left\{\left[\sum_{j=1-x}^{n} \frac{j+f/d}{\omega} gv^{j+f/d}\right] + \frac{n+f/d}{\omega}100v^{n+f/d}\right\} &(7)\\ &= \frac{v^{f/d}}{\omega P}\left\{g\left[\sum_{j=1-x}^{n}(j+\frac{f}{d})v^j\right] + 100(n+\frac{f}{d})v^n\right\} \\ &= \frac{v^{f/d}}{\omega P}\left\{g\left[\left(\sum_{j=1-x}^{n} jv^j\right) + \frac{f}{d}\left(\sum_{j=1-x}^{n} v^j\right)\right] + 100(n+\frac{f}{d})v^n\right\} &(8)\end{aligned}$$

Let

$$S' \equiv \sum_{j=1}^{n} jv^j \tag{9}$$

Then,

$$\begin{aligned} S' &= v + 2v^2 + 3v^3 + ... + nv^n \\ vS' &= v^2 + 2v^3 + 3v^4 + ... + nv^{n+1} \\ (1-v)S' &= v + v^2 + v^3 + ... + v^n - nv^{n+1} \\ &= S - nv^{n+1} \\ &= \frac{1-v^n}{i} - nv^{n+1} \text{ by substituting (4)} \end{aligned}$$

[1] Since if $x = 0$ (ex-interest) then

$$\sum_{j=1-x}^{n} gv^j = \sum_{j=1}^{n} gv^j = S$$

and if $x = 1$ (cum-interest) then

$$\sum_{j=1-x}^{n} gv^j = \sum_{j=0}^{n} gv^j = 1 + S$$

$$
\begin{aligned}
S' &= \frac{1}{iv}\left(\frac{1-v^n}{i} - nv^{n+1}\right) \\
&= \frac{1}{i}\left(\frac{1-v^n}{iv} - nv^n\right) \qquad (10)
\end{aligned}
$$

Fortunately[2],

$$\sum_{j=1-x}^{n} jv^j = \sum_{j=1}^{n} jv^j = S'$$

Hence we substitute (10) and (5) into (8) to give

$$D = \frac{v^{f/d}}{\omega P}\left\{g\left[\frac{1}{i}\left(\frac{1-v^n}{iv} - nv^n\right) + \frac{f}{d}\left(x + \frac{1-v^n}{i}\right)\right] + 100(n + f/d)v^n\right\} \qquad (11)$$

# 4 Relationship between Duration, Modified Duration and Price

The bond duration $D$ can be expressed in terms of the partial derivative of the bond price $P$ with respect to the coupon period yield $i$ ($\partial P/\partial i$).

From (1) we obtain

$$
\begin{aligned}
\frac{\partial P}{\partial i} &\equiv \left[\sum_{j=1-x}^{n} -(j + \frac{f}{d})gv^{j+f/d+1}\right] - (n + \frac{f}{d})100v^{n+f/d+1} \qquad (12) \\
&= -\omega vP\left\{\frac{1}{P}\left[\sum_{j=1-x}^{n} g\frac{j + f/d}{\omega}v^{j+f/d}\right] + \frac{n + f/d}{\omega}100v^{n+f/d}\right\} \qquad (13)
\end{aligned}
$$

Substituting (7) into (13) gives

$$\frac{\partial P}{\partial i} = -\omega vPD$$

Hence,

$$D = -\frac{1}{\omega vP}\frac{\partial P}{\partial i} \qquad (14)$$

The Modified (Hick's) Duration is defined by

$$D^\star \equiv \frac{D}{1+i} = Dv \qquad (15)$$

Hence, by substituting (11) into (15),

$$D^\star = -\frac{1}{\omega P}\frac{\partial P}{\partial i} \qquad (16)$$

[2] If $x = 1$ (cum-interest), then

$$\sum_{j=0}^{n} jv^j = 0v^0 + 1v^1 + ... + nv^n = v + 2v^2 + ... + nv^n = S'$$

## 5 Bond Convexity

Bond convexity is defined by

$$C \equiv \frac{1}{\omega^2 P}\frac{\partial^2 P}{\partial i^2} \tag{17}$$

The second partial derivative of $P$ with respect to $i$ can be obtained by partial differentiating (12)

$$\frac{\partial^2 P}{\partial i^2} = \left[\sum_{j=1-x}^{n}(j+\frac{f}{d})(j+\frac{f}{d}+1)gv^{j+f/d+2}\right] + (n+\frac{f}{d})(n+\frac{f}{d}+1)100v^{n+f/d+2} \tag{18}$$

Expanding $(j+\frac{f}{d})(j+\frac{f}{d}+1)$ gives

$$\begin{aligned}(j+\frac{f}{d})(j+\frac{f}{d}+1) &= j^2+(\frac{f}{d}+1)j+\frac{f}{d}j+\frac{f}{d}(\frac{f}{d}+1)\\ &= j^2+(2\frac{f}{d}+1)j+\frac{f}{d}(\frac{f}{d}+1)\end{aligned} \tag{19}$$

Substituting (19) into (18) gives

$$\begin{aligned}\frac{\partial^2 P}{\partial i} &= \left\{\sum_{j=1-x}^{n}\left[j^2+(2\frac{f}{d}+1)j+\frac{f}{d}(\frac{f}{d}+1)\right]gv^{j+f/d+2}\right\}\\ &\quad +(n+\frac{f}{d})(n+\frac{f}{d}+1)100v^{n+f/d+2}\\ &= gv^{2+f/d}\left\{\left[\sum_{j=1-x}^{n}j^2v^j\right]+(2\frac{f}{d}+1)\left[\sum_{j=1-x}^{n}jv^j\right]+\frac{f}{d}(\frac{f}{d}+1)\left[\sum_{j=1-x}^{n}v^j\right]\right\}\\ &\quad +(n+\frac{f}{d})(n+\frac{f}{d}+1)100v^{n+f/d+2}\end{aligned} \tag{20}$$

Let

$$S^{\star} \equiv \sum_{j=1}^{n} j^2 v^j \tag{21}$$

Then,

$$\begin{aligned}S^{\star} &= v+4v^2+9v^3+...+n^2v^n\\ vS^{\star} &= v^2+4v^3+9v^4+...+n^2v^{n+1}\\ (1-v)S^{\star} &= v+3v^2+5v^3+...+[n^2-(n-1)^2]v^n-n^2v^{n+1}\\ &= v+3v^2+5v^3+...+(2n-1)v^n-n^2v^{n+1}\end{aligned} \tag{22}$$

Now, let

$$S^{\square} \equiv \sum_{j=1}^{n}(2j-1)v^j \tag{23}$$

Then,

$$S^{\square} = v+3v^2+5v^3+...+(2n-1)v^n$$

$$
\begin{aligned}
vS^{\square} &= v^2 + 3v^3 + 5v^4 + ... + (2n-1)v^{n+1} \\
(1-v)S^{\square} &= v + 2v^2 + 2v^3 + ...2v^n - (2n-1)v^{n+1} \\
&= 2(v + v^2 + v^3 + ...v^n) - v - (2n-1)v^{n+1} \\
&= 2S - v - (2n-1)v^{n+1} \\
&= \frac{2(1-v^n)}{i} - v - (2n-1)v^{n+1} \text{ by substituting (4)} \\
S^{\square} &= \frac{1}{iv}\left[\frac{2(1-v^n)}{i} - v - (2n-1)v^{n+1}\right] \qquad (24)
\end{aligned}
$$

Substitute (24) back into (22) gives

$$
\begin{aligned}
(1-v)S^{\star} &= \frac{1}{iv}\left[\frac{2(1-v^n)}{i} - v - (2n-1)v^{n+1}\right] - n^2v^{n+1} \\
S^{\star} &= \frac{1}{i^2v^2}\left[\frac{2(1-v^n)}{i} - v - (2n-1)v^{n+1}\right] - \frac{1}{iv}n^2v^{n+1} \\
S^{\star} &= \frac{1}{i^3v^2}\left[2(1-v^n) - iv - (2n-1)iv^{n+1} - n^2i^2v^{n+2}\right] \qquad (25)
\end{aligned}
$$

Again, note that

$$\sum_{j=1-x}^{n} j^2v^j = \sum_{j=1}^{n} j^2v^j = S^{\square}$$

Substitute (25), (10) and (5) into (20) gives

$$
\begin{aligned}
\frac{\partial^2 P}{\partial i} &= gv^{2+f/d}\left\{\frac{1}{i^3v^2}\left[2(1-v^n) - iv - (2n-1)iv^{n+1} - n^2i^2v^{n+2}\right]\right. \\
&\quad \left. +(2\frac{f}{d}+1)\frac{1}{i}\left[\frac{1-v^n}{iv} - nv^n\right] + \frac{f}{d}(\frac{f}{d}+1)\left[x + \frac{1-v^n}{i}\right]\right\} \\
&\quad +(n+\frac{f}{d})(n+\frac{f}{d}+1)100v^{n+f/d+2} \qquad (26)
\end{aligned}
$$

Putting (26) back into (17) gives

$$
\begin{aligned}
C &= \frac{v^{2+f/d}}{\omega^2 P}\left\{g\left[\frac{1}{i^3v^2}\left(2(1-v^n) - iv - (2n-1)iv^{n+1} - n^2i^2v^{n+2}\right)\right.\right. \\
&\quad \left. +(2\frac{f}{d}+1)\frac{1}{i}\left(\frac{1-v^n}{iv} - nv^n\right) + \frac{f}{d}(\frac{f}{d}+1)\left(x + \frac{1-v^n}{i}\right)\right] \\
&\quad \left. +(n+\frac{f}{d})(n+\frac{f}{d}+1)100v^n\right\} \qquad (27)
\end{aligned}
$$

# 6 Dispersion, and its relationship with Duration and Convexity

The dispersion of a bond $M^2$ is defined to be the sum of present values of all future cash flows weigthed by the square of the years of maturity of each cash flow and subtracted by the square of the bond duration. We can express the definition of bond dispersion as

$$M^2 = \frac{1}{P}\left\{\left[\sum_{j=1-x}^{n}\left(\frac{j+f/d}{\omega}\right)^2 gv^{j+f/d}\right] + \left(\frac{n+f/d}{\omega}\right)^2 100v^{n+f/d}\right\} - D^2 \qquad (28)$$

The dispersion of a bond is related to bond duration and bond convexity. From 17, we have

$$\begin{aligned} C &= \frac{1}{\omega^2 P}\left\{\left[\sum_{j=1-x}^{n}(j+\frac{f}{d})(j+\frac{f}{d}+1)gv^{j+f/d+2}\right]+(n+\frac{f}{d})(n+\frac{f}{d}+1)100v^{n+f/d+2}\right\} \\ &= \frac{v^2}{\omega^2 P}\left\{\left[\sum_{j=1-x}^{n}\left((j+\frac{f}{d})^2+(j+\frac{f}{d})\right)gv^{j+f/d}\right]+\left[(n+\frac{f}{d})^2+(n+\frac{f}{d})\right]100v^{n+f/d}\right\} \\ &= v^2\left\{\frac{1}{P}\left(\left[\sum_{j=1-x}^{n}\left(\frac{j+f/d}{\omega}\right)^2 gv^{j+f/d}\right]+\left(\frac{n+f/d}{\omega}\right)^2 100v^{n+f/d}\right)-D^2\right. \\ &\quad \left.+D^2+\frac{1}{\omega}\frac{1}{P}\left(\left[\sum_{j=1-x}^{n}\frac{j+f/d}{\omega}gv^{j+f/d}\right]+\frac{n+f/d}{\omega}100v^{n+f/d}\right)\right\} \end{aligned} \tag{29}$$

Subsituting (28) and (7) into (29) gives

$$C = v^2\left(M^2+D^2+\frac{D}{\omega}\right)$$

Hence,

$$M^2 = \frac{C}{v^2}-\frac{D}{\omega}-D^2 \tag{30}$$